# Section 9.4 part 2

approach to the First Sylow theorem

9.4 Proof of Sylow Theorems

**Recall** Action of $G$ (group) on $S$ (set). (Left)

- homomorphism $G \to A(S)$ $\quad A(S) = \{ f : S \to S \mid f \text{ is a bijection} \}$

group operation - composition of functions

Notation: $g \mapsto f_g$

$f_g : S \to S$

$x \mapsto f_g(x)$

We write

$g \cdot x = f_g(x) \quad g \in G,\ x \in S$

We have $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ $\quad f_{g_1 g_2}(x) = f_{g_1}(f_{g_2}(x))$

- the map $G \to A(S)$ is a group homomorphism

**Orbit of $x$**: $\quad Orb(x) = \{ g \cdot x \mid g \in G \} \subseteq S$

The relation on $S$ define by $x \sim y$ iff $x$ and $y$ are on the same orbit is an equivalence relation

that is $y = g \cdot x$ for some $g \in G$

$g^{-1} y = x$

$S = \bigcup_{x \in S} Orb(x)$ - partition of $S$ into non-overlapin subsets - equivalence classes with respect to the relation $\sim$ on $S$

For an element $x \in S$ we define the stabilizer of $x$:

$$St(x) = \{ g \in G \mid g \cdot x = x \} \subseteq G$$

For any $x \in S$, $St(x)$ is a subgroup of $G$

Pf (using Th7.11)

$g_1, g_2 \in St(x)$

$(g_1 g_2) \cdot x = g_1 (g_2 \cdot x)$

$= g_1 \cdot x = x$ $\quad g_1 g_2 \in St(x)$

$g \in St(x)$

$g^{-1} \cdot x = g^{-1} \cdot (g \cdot x)$

$= (g^{-1} g) \cdot x = e \cdot x = x$

$g^{-1} \in St(x)$

$$G = \bigcup_{a \in G} a\, St(x)$$ - the group is partitioned into left cosets

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Relation between $Orb(x) \subseteq S$ and $St(x) \subseteq G$

$Orb(x) = \{ g \cdot x \mid g \in S \} = \{ as \cdot x \mid s \in St(x) \}$, $a$ - a representative of a left coset

$as \cdot x = a(s \cdot x) = a \cdot x$

Choose one representative $a \in G$ for every coset in $G = \bigcup_a a\, St(x)$

$= \{ a \cdot x \mid a - \text{the representative} \}$

There are exactly as many elements in $Orb(x)$ as there are cosets

$[G : St(x)]$

**Prop** Let $x \in S$.

Assume that either the set $Orb(x) \subseteq S$ or the index $[G : St(x)]$ is finite. Then so is another one, and we have that

$$|Orb(x)| = [G : St(x)]$$

When $G$ is finite, we have

$$[G : St(x)] = |G| / St(x)$$

$$|Orb(x)| = |G| / St(x)$$

In particular, $|Orb(x)| \,\big|\, |G|$

1-st application - "class equation"

$G$ - a finite group. Take $S = G$.

Action: $g \cdot x = gxg^{-1}$

$\mathrm{Orb}(x) = C_x = \{ gxg^{-1} \mid g \in G \}$ - conjugacy class of $x \in G$

$St(x) = C(x) = \{ g \in G \mid x = g \cdot x \}$

$= \{ g \in G \mid x = gxg^{-1} \}$

$= \{ g \in G \mid xg = gx \}$ - centralizer of $x \in G$

- all elements of $G$ which commute with $x$

$$|C_x| = [G : C(x)]$$

Partition of the set $G$ into conjugacy classes (orbits)

$$G = \bigcup_{x \in G} C_x$$

Prop. This is indeed an action of $G$ on the set $G$.

Pf

The map $G \to G$, $x \mapsto gxg^{-1}$ is a bijection for every $g \in G$.

- this is an inner automorphism; the inverse is $y \mapsto g^{-1}yg$

$(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$:

$(g_1 g_2) \cdot x = g_1 g_2 x (g_1 g_2)^{-1}$

$= g_1 g_2 x g_2^{-1} g_1^{-1} = g_1 (g_2 \cdot x) g_1^{-1}$

$= g_1 \cdot (g_2 \cdot x)$

Let $b_1, \ldots, b_s$ be representatives, one in every conjugacy class

$G = C_{b_1} \cup C_{b_2} \cup \dots \cup C_{b_s}$, distinct classes do not overlap

$|G| = |C_{b_1}| + |C_{b_2}| + \dots + |C_{b_s}|$ $\Big\}$ $|C_b| = [G : C(b)]$

Work out the case when $|C_b| = [G : C(b)] = 1$

$$G = C(b)$$

$$G = \{g \in G \mid gb = bg\}$$

$|C_b| = 1$ (i.e. $C_b = \{b\}$) iff $b$ commutes with any element of the group

Recall: center of $G$ $Z(G) = \{c \in G \mid gc = cg$ for every $g \in G\}$

$Z(G)$ is an abelian subgroup in $G$

$|C_b| = 1$ iff $b \in Z(G)$

$|G| = |Z(G)| + |C_1| + \dots + |C_t|$ with all $|C_i| \mid |G|$

(3), p306

class equation
class formula

First Sylow Theorem is derived from this (p.307) - induction in $|G|$